Admissions Exam for Regular and Integrated MSc-PhD
Computer Science
Ashoka University
Sample
Time: 2 Hours

Instructions

PLEASE READ CAREFULLY BEFORE PROCEEDING

- This is a open book, open Internet exam. You are free to consult any resources as you see fit. However, whenever you consult any resource(s), you need to cite your source, otherwise it would be considered as plagiarised.
- This is an **individual** exam. You are **not allowed** to consult any other person during the exam. All your answers should be written by you only. Any evidence of academic dishonesty plagiarism, or copying from Internet and other sources without citations will lead to disqualification.
- This exam contains **24** questions in all. Please answer as many questions as you can.
- Part 1 of this exam is for ALL candidates i.e., candidates with Computer Science (CS) background as well as candidates from non-CS backgrounds.
- Part 2 of this exam is specifically for candidates with CS background.
- Please write your answers in LaTeX, Microsoft Word, Google Docs, or any other word processor of your choice. Scanning hand-written text and converting to PDF is also allowed, as long as the answers are clearly readable/understandable.
- Your final answer script should be a single PDF file.
- Make sure to put your Name and Application number at the top of the answer sheet.
- Please arrange your answers in the order of questions that have been asked.
- Please name your submission file as per the following convention : `firstName_lastName_applicationNumber.pdf`. The `firstName` and `lastName` should be replaced by your first and last names, respectively. `applicationNumber` should be replaced by the application number of your PhD application.
- A 10 min buffer is being provided to overcome any Internet-related issues that you might have. Submit your answers positively by **4.10 pm**. Answer scripts submitted after this will NOT be evaluated.
- Marks for each question are indicated in parenthesis.

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Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Score																

Problem	17	18	19	20	21	22	23	24
Score								

Part 1: For all candidates

1. Let $\{X_i\}_{i=1}^{2n}$ is a collection of random variables each following Bernoulli distribution, i.e., X_i takes values in $\{0,1\}$ with $\mathbb{P}[X_i = 0] = p$ and $\mathbb{P}[X_i = 1] = 1 - p$ where $0 \leq p \leq 1$. Consider the following random variable

$$Y = (X_1 - X_{n+1}) + (X_2 - X_{n+2}) + \cdots + (X_n - X_{2n})$$

Determine the value of $\mathbb{P}[Y = n + 1]$ **(3 points)**

2. Let X, Y be random variables taking values in the set $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ respectively. Determine the value of

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j$$

where $p_i = \mathbb{P}[X = a_i]$ and $q_j = \mathbb{P}[Y = b_j]$. **(3 points)**

3. Let X be a random variable taking values in the set $S = \{1, \ldots, 10\}$ with uniform distribution. Consider the following random variable $Y = 2X$. Determine the value of

$$\sum_{i=1}^{20} (\mathbb{P}[Y = i] - \frac{1}{20})$$

(3 points)

4. Suppose A and B are independent random variables, each uniform on $(0, 1)$. Let C be the smaller of the two.
   (a) Show the region $C > x$. What is the probability $P[C > x]$?
   (b) Plot the density function and the CDF for C in the interval $(-1, 1)$.

(4 points)

5. Consider a cultural club, which offers two types of activities, namely yoga and dancing. Each member of the club must register for at least one of these activities. The club has 100 members. 43 members have registered for dancing. 23 members are registered for both yoga and dancing. Determine the number of members, who have registered for only yoga, but not registered for dancing. **(3 points)**

6. Suppose John is at a rifle shooting range. Out of every 6 shots he takes, he is able to hit the target once. John takes 6 consecutive shots in the rifle shooting range. Determine the probability that John will hit the target. **(3 points)**

7. Ashoka University has four troublesome departments which have 7, 37, 13 and 5 faculty members. The VC wishes to form a faculty disciplinary committee (DISCO) to hear cases of student misbehaviour. To avoid the possibility of ties, the DISCO will have three members. To avoid favouritism the DISCO members will be from different departments and the DISCO will change daily. If the DISCO only sits during the normal academic year (165 days), how many years can pass before a DISCO must be repeated? **(4 points)**

8. We want to know how many ways n boys and $n + 1$ girls can sit in a row at a table.
   (a) How many ways can this be done if there are no restrictions?

(b) How many ways can this be done if the boys sit together and the girls sit together?

(c) How many ways can this be done if the boys and girls must alternate?

What are the answers if the table is circular? **(6 points)**

9. Melon Usk is hosting a party for his employees at the bicycle company Nikola. He has invited n people in total to the event, many of whom are already friends. In fact, there are at least $n^2/4$ pairs of people who are already friends with each other.

   Mr. Usk wants to end the party by having a popular person sing "So Long, Farewell" from The Sound of Music. We define a "popular" person as someone who is friends with at least $n/8$ people at the party.

   (a) Prove that there must be at least $3n/7$ popular guests at the party. **(5 points)**

   (b) Mr. Usk asks you to find a popular person. You decide to repeat the following strategy: ask a random guest how many friends they have, and if they are popular, then get them to sing. Prove that with at least 99% probability, you will find a popular guest after asking only a constant number of people. **(6 points)**

10. Argue, as formally as possible, that if $X \subset \mathbb{R}$ is closed and bounded and $f : X \to \mathbb{R}$ is continuous, then f is bounded and attains a maximum and a minimum. **(4 points)**

11. Let A, B, C, D, E be $n \times n$ real matrices such that

$$A \times \left[B \parallel C\right] = \left[D \parallel E\right],$$

where the operation $\parallel$ denotes concatenation of two matrices. Thus, the matrix $\left[B \parallel C\right]$ is of order $n \times 2n$. If $E = \mathbb{I}_n$, the $n \times n$ identity matrix, and

$$C = \begin{bmatrix} 100 & 0 & \cdots & 0 \\ 0 & 100 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 100 \end{bmatrix}$$

, then determine the matrix D. **(5 points)**

12. (a) Show that a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ has full rank if and only if it maps no two distinct vectors to the same vector.

   (b) A collection of subspaces $S_1, \ldots, S_p$ in $\mathbb{R}^m$ is *mutually orthogonal* if $\mathbf{x}^T\mathbf{y} = 0$ whenever $\mathbf{x} \in S_i$ and $\mathbf{y} \in S_j$ for $i \neq j$. The *orthogonal complement* of a subspace $S \subseteq \mathbb{R}^m$ is defined by

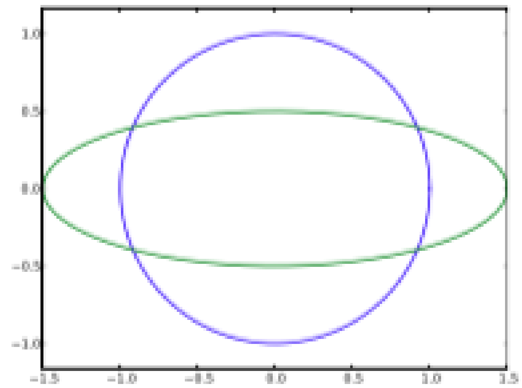
$$S^\perp = \{\mathbf{y} \in \mathbb{R}^m \ : \ \mathbf{y}^T\mathbf{x} = 0 \ \ \forall \mathbf{x} \in S\}$$

   Show that for any matrix $\mathbf{A}$

$$range(\mathbf{A})^\perp = null(\mathbf{A}^T)$$

(5 points)

13. Consider the plot of the unit circle in $\mathbb{R}^2$ and a plot of $\{Av\}$, where A is a 2×2 matrix and v is a vector on the unit circle. Read off from the plot the 2-norm of the matrix. Is it possible to read off the matrix itself from the plot?



(4 points)

14. Suppose

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} -0.0776 & -0.8331 & 0.5456 & -0.0478 \\ -0.3105 & -0.4512 & -0.6919 & 0.4704 \\ -0.5433 & -0.0694 & -0.2531 & -0.7975 \\ -0.7762 & 0.3124 & 0.3994 & 0.3748 \end{bmatrix} \begin{bmatrix} -12.8841 & -14.5916 & -16.2992 \\ 0 & -1.0413 & -2.0826 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a **QR** factorisation of **A**. Compute the *rank*, *nullity*, *null space* and *range space* of both **A** and $\mathbf{A}^T$. **(6 points)**

15. Suppose the flow (f) of water from a tap at time t is described by the following function:

$$f = 25 + 22t–10t^2 \tag{1}$$

Determine the time at which the flow of water from the tap is maximum, and the value of flow (f) corresponding to that time. **(3 points)**

16. Find the maximum product of two numbers, whose sum equals 24. **(3 points)**

Part 2: For candidates with Computer Science background

17. You have a computer with 8 bits of memory in total. You are asked to compute, for an input n, the value 2^{2^n}. What is the largest n you can handle? **(2 points)**

18. In any list, let us call a number that is bigger than both of its neighbours a "nail". Given a list of unique positive numbers, we append 0 to both sides. Show that there must exist a nail in any such list. **(4 points)**

19. You have a recursive program which takes a problem of size n and, with only $7n$ work, produces three subproblems: one of size $\frac{n}{3}$, another of size $\frac{n}{6}$, and a third, of size $\frac{n}{9}$.

    Can you show that the algorithm runs in time $T(n) \leq 18n$? For full credit, make sure to first state the recurrence equation. **(4 points)**

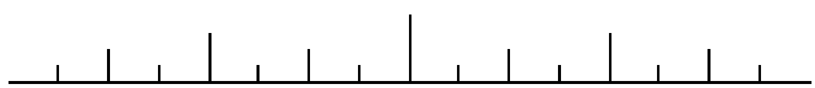
20. We need to write code for a second price auction, that is, an auction where the second highest bidder wins. That is, given a list of bidders and corresponding bids, extract the second highest

bidder in worst case $O(\log n)$ time. You can assume you already have a previously implemented version of a max-priority queue. **(4 points)**

21. Suppose your favourite programming language can only perform an add (`add`: $\mathbb{W} \times \mathbb{W} \rightarrow \mathbb{W}$, where $\mathbb{W}$ is the set of Whole numbers) operation, but not multiply.
    (a) Then, define multiplication (`mul`: $\mathbb{W} \times \mathbb{W} \rightarrow \mathbb{W}$) in terms of the `add`: $\mathbb{W} \times \mathbb{W} \rightarrow \mathbb{W}$ function (`add(a,b)` returns $a + b$). How many steps does your algorithm take?
    (b) Suppose we include, in addition to `add`, functions `dec` : $\mathbb{W}^+ \rightarrow \mathbb{W}$, which decrements an integer by 1, `even` : $\mathbb{W} \rightarrow \{true, false\}$, which tests whether or not a given number is even, `double` : $\mathbb{W} \rightarrow \mathbb{W}$, which doubles an integer, and `halve` : $\mathbb{W} \rightarrow \mathbb{W}$, which divides an (even) integer by 2. Can you construct a more efficient algorithm for `mult`?

    (5 points)

22. Write an algorithm (pseudo-code) to draw markings on a ruler. Given a parameter n signifying a resolution of $1/2^n$, you have to put a mark at every integral point between 0 and 2^n, endpoints not included. The middle mark should be n units high, the marks in the middle of the left and right halves should be $n-1$ units high, and so on (see the figure for $n = 4$).



    Assume that a procedure called $mark(x, h)$ is available which puts a mark of height h at location x.

    How many steps do you require? Can you argue that your algorithm is correct? **(6 points)**

23. Let us define the integer square root of n as the integer k such that $k^2 \leq n < (k+1)^2$. The integer square root can be computed using the following inductive process:
    (a) Compute the integer square root i of $m = n \ div\ 4$ by induction hypothesis. We then have that $i^2 \leq m < (i+1)^2$.
    (b) Since m and i are integers we have that $(m+1) \leq (i+1)^2$. We thus have $(2i)^2 \leq 4m \leq n < 4m + 4 \leq (2i+2)^2$. Hence, for the induction step, we have that the integer square root of n is either $2i$ or $2i+1$.

    Write a pseudo-code for an algorithm based on the above inductive reasoning. **(6 points)**

24. Suppose you are given an array containing positive and negative integers. How would you *efficiently* separate out the positive and negative numbers so that you can output the positive numbers first, and then you can output the negative numbers? **(4 points)**