Admissions Exam for Regular and Integrated MSc-PhD
Computer Science
Ashoka University
Sample
Time: 2 Hours

Instructions

PLEASE READ CAREFULLY BEFORE PROCEEDING

- This is a open book, open Internet exam. You are free to consult any resources as you see fit. However, whenever you consult any resource(s), you need to cite your source, otherwise it would be considered as plagiarised.
- This is an **individual** exam. You are **not allowed** to consult any other person during the exam. All your answers should be written by you only. Any evidence of academic dishonesty plagiarism, or copying from Internet and other sources without citations will lead to disqualification.
- This exam contains **21** questions in all. Please answer as many questions as you can.
- Part 1 of this exam is for ALL candidates i.e., candidates with Computer Science (CS) background as well as candidates from non-CS backgrounds.
- Part 2 of this exam is specifically for candidates with CS background.
- Please write your answers in LaTeX, Microsoft Word, Google Docs, or any other word processor of your choice. Scanning hand-written text and converting to PDF is also allowed, as long as the answers are clearly readable/understandable.
- Your final answer script should be a single PDF file.
- Make sure to put your Name and Application number at the top of the answer sheet.
- Please arrange your answers in the order of questions that have been asked.
- Please name your submission file as per the following convention : `firstName_lastName_applicationNumber.pdf`. The `firstName` and `lastName` should be replaced by your first and last names, respectively. `applicationNumber` should be replaced by the application number of your PhD application.
- A 10 min buffer is being provided to overcome any Internet-related issues that you might have. Submit your answers positively by **4.10 pm**. Answer scripts submitted after this will NOT be evaluated.
- Marks for each question are indicated in parenthesis.

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Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Score																

Problem	17	18	19	20	21
Score					

Part 1: For all candidates

1. Prove the following using the stated methods:

   (a) By contraposition: If n is a positive integer with $n^2 > 100$, then $n > 10$. **(5 points)**

   (b) By induction: Every integer greater than 1 is a product of prime numbers. **(5 points)**

2. (a) Show that composition of functions is associative. **(5 points)**

   (b) Prove that among any five points in the plane with integer coordinates, there are at least two points such that the center of the line segment that connects them also has integer coordinates. **(5 points)**

3. (a) A doctor gives a patient a test for a particular cancer. Before the results of the test, the only evidence the doctor has to go on is that 1 woman in 1000 has this cancer. Experience has shown that, in 99 percent of the cases in which cancer is present, the test is positive; and in 95 percent of the cases in which it is not present, it is negative. If the test turns out to be positive, what probability should the doctor assign to the event that cancer is present?

   How large must the prior probability of cancer be to give a posterior probability of 0.5 for cancer given a positive test? **(6 points)**

   (b) You are given two urns and fifty balls. Half of the balls are white and half are black. You are asked to distribute the balls in the urns with no restriction placed on the number of either type in an urn. How should you distribute the balls in the urns to maximize the probability of obtaining a white ball if an urn is chosen at random and a ball drawn out at random? Justify your answer. **(5 points)**

4. Let Σ be the alphabet $\{a, b, c\}$. Show that the number of words of length n in which the letter a appears an even number of times is

$$(3^n + 1)/2$$

   You may use induction or any other proof technique. **(10 points)**

5. (a) Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a \ \rho \ a'$ if and only if $f(a) \ \sigma \ f(a')$. Prove that ρ is an equivalence relation on A. **(5 points)**

   (b) If p is a prime, prove that $p^2 - 1$ is divisible by 24 if $p > 3$. **(5 points)**

6. 11 men and 8 women are to be seated such that no two women sit together. In how many ways, can this be done, if:

   (a) They are seated in a straight row. **(5 points)**

   (b) They are seated around a circular table. **(5 points)**

7. An array is called a palindrome if it has the same sequence of numbers when its elements are traversed both backwards and forwards. For example,

   `a[0..3] = [1,2,2,1]` is a palindrome
   `b[0..4] = [3,4,5,4,3]` is a palindrome
   `c[0..3] = [1,2,3,2]` is not a palindrome

`c[0..0] = [3]` is a palindrome

Write an algorithm to check if a given array `x[0..n-1]` is a palindrome. **(5 points)**

8. This problem involves starting out with an array `A[0..n-1]` of n_r red elements, n_w white elements and n_b blue elements in random order such that $0 \leq n_r, n_w, n_b \leq n$ and $n_r + n_w + n_b = n$, and arranging them such that all reds occur before all whites which occur before all blue. The re-arrangement has to achieved by making only one pass over the array using only swap operations. Give an algorithm for solving the problem. You may assume that the red, white and blue elements are represented by integers 1, 2 and 3 respectively. **(5 points)**

9. The *integer square root* of an integer n is the integer k such that $k^2 \leq n < (k+1)^2$, It can be computed using the following inductive process:

   - Compute the integer square root i of $m = n \; div \; 4$ recursively. We then have that $i^2 \leq m < (i+1)^2$
   - Since m and i are integers we have that $(m+1) \leq (i+1)^2$. We thus have $(2i)^2 \leq 4m \leq n < 4m + 4 \leq (2i+2)^2$. Hence we have that the integer square root of n is either $2i$ or $2i+1$.

   (a) Write a recursive program corresponding to the above algorithm, argue its correctness and estimate its time complexity. **(5 points)**

   (b) Write an iterative version corresponding to the above algorithm. Describe the *invariant* for the iterative process and derive the number of steps required. **(5 points)**

10. Let $k, n \in \mathbb{N}$. Define $A_{k,n} = \{w \in \{-1, 0, 1\}^n \mid \mathsf{wt}(w) = k\}$, where $\mathsf{wt}(w)$ counts the number of non-zero entries in the vector w. Determine the size of the set $A_{k=10,n=11}$.

    (5 points)

11. Let a, b be two positive integers. Let $\mathsf{Bin}(a) = (c_0, c_1, \ldots, c_{k-2}, c_{k-1})$ denote the binary representation of a, i.e., $c_i \in \{0, 1\}$ and $\sum_{i=0}^{k-1} c_i 2^i = a$. Let $\mathsf{power}_k(b) = (b, 2b, \ldots, 2^{k-2}b, 2^{k-1}b)$. Determine the value of the inner product $\langle \mathsf{Bin}(a), \mathsf{power}_k(b) \rangle$, where inner product of two vectors $\langle x, y \rangle = x \cdot y^T$. y^T denotes the transpose of the row vector y.

    (5 points)

12. Consider the matrix $\mathbf{A}$ such that:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 7 & 2 & 1 & 0 \\ 10 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 & 10 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Compute the *rank*, *nullity*, *null space* and *range space* of both $\mathbf{A}$ and $\mathbf{A}^T$. **(5 points)**

(b) Consider right hand sides $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{A}^T\mathbf{x} = \mathbf{c}$. Derive conditions under which the systems have solutions and indicate what the solutions are if the conditions are satisfied. **(5 points)**

13. Suppose $\mathbf{A}$ is a 2×1 matrix and that $\mathbf{B}$ is a 1×2 matrix. Prove that $\mathbf{C} = \mathbf{AB}$ is not invertible. **(4 points)**

14. Consider the following systems of equation and find out whether they have solutions. If so, explicitly describe all solutions.

(a)

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & & & + & 2x_3 & = & 1 \\ x_1 & - & 3x_2 & + & 4x_3 & = & 2 \end{array}$$

(5 points)

(b)

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & = & 1 \\ x_1 & + & x_2 & - & x_3 & + & x_4 & = & 2 \\ x_1 & + & 7x_2 & - & 5x_3 & - & x_4 & = & 3 \end{array}$$

(5 points)

Part 2: For candidates with CS background

15. Write only the answers of the following questions. No need to write any explanation.

(a) Write the asymptotic upper bound of $T(n) = T(3n/10) + T(7n/10) + 10n$. **(2 points)**

(b) The array representing a max-heap is:

20	12	10	8	4	7	6	5	2	3

Write the value of the element stored in the parent node of the node containing the smallest element, that is, '2'. **(2 points)**

(c) Given an unordered list L of n integers, what is a tight lower bound on the worst case time complexity to find a/the pair of elements from L, which have the minimum difference among all possible pairs in L? **(2 points)**

(d) What is a tight lower bound on the average-case time complexity for successful search in a complete binary tree (which is not necessarily a binary search tree). **(2 points)**

(e) An undirected graph G containing n nodes and $O(n \log n)$ edges is represented by an adjacency matrix. Write the time complexity that will be required by an optimal algorithm to prepare the adjacency list of G from its adjacency matrix. **(2 points)**

16. You are to implement a hash table to store at most 1000 integers. The hash function used for any key k is $k\%1000$. The collision resolution scheme used is open addressing.

(a) Define a datatype `HASHTABLE` to represent a hash table to store at most `1000` integers. **(2 points)**

(b) Write a C function `int Insert(HASHTABLE H, int k)` that inserts a key `k` into a hashtable `H`. The function returns `1` if the insert is successful, `0` if the table is full. **(5 points)**

(c) Write a C function `int Delete(HASHTABLE H, int k)` that deletes a key `k` from a hashtable `H`. The function returns `1` if `k` is found in the table and deleted, `0` if `k` is not found in the table. **(3 points)**

17. A $k-ary\ tree$ is a rooted tree in which each node has no more than k children. A $perfect\ k-ary\ tree$ is a $k-ary\ tree$ where every node has either 0 or k children, and all leafs are at the same depth.

(a) Define a C data structure `KNode` to store one node of a $k-ary\ tree$ for some given constant k. Assume that the tree stores one integer value at each node. **(3 points)**

(b) Write a *recursive* C function `IsPerfect()` that will take the pointer `T` to the root of a $k-ary\ tree$ as parameter and return `1` if the tree is a $perfect\ k-ary\ tree$, `0` otherwise. You are free to design the prototype of the function. Do not use any global variables. **(7 points)**

18. (a) Give a context free grammar for the following language L_4. The alphabet Σ is $\{0, 1, \#\}$. $L_4 = \{w : w$ starts and ends with the same symbol, the length is odd, and the middle symbol is $\#\}$.

Draw the parse tree of the following string with the above grammar:

$$0101\#1110$$

(5 points)

(b) Prove that if L is context free, then L^* must be context free. **(5 points)**

19. (a) Implement the four Boolean functions listed using three half-adder circuits:

$$\begin{aligned} D &= A \bigoplus B \bigoplus C \\ E &= A'BC + AB'C \\ F &= ABC' + (A' + B')C \\ G &= ABC \end{aligned}$$

(5 points)

(b) Design a sequential circuit with JK flip-flops to satisfy the following state equations:

$$\begin{aligned} A(t+1) &= A'B'CD + A'B'C + ACD + AC'D' \\ B(t+1) &= A'C + CD' + A'BC' \end{aligned}$$

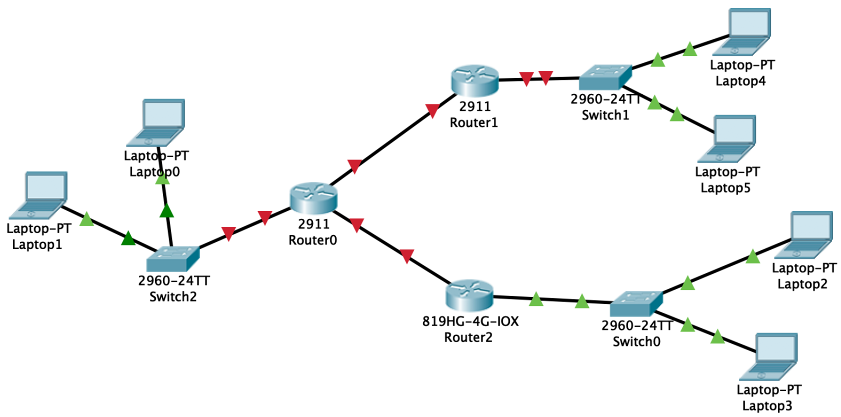
(5 points)

20. You're in charge of the new Digital Election Commission of India. Due to privacy and security reasons, you are not allowed to know which election (district, municipality, etc.) you're doing the counting for, so you don't know the population size or the number of candidates beforehand. Unfortunately, you're not very well funded, so you have very limited storage for each machine – just some constant amount which is much lesser than the number of votes, and possibly even lesser than the number of candidates, which may both be arbitrary.

So you get a sequence of votes: "Amar, Amar, Akbar, Anthony, Amar, ..." – and whenever you hit "DONE!" (a special command given by the local officer), the sequence is finished. You don't know when the sequence will end; you just get one name after the other. Your job is to simply figure out if someone won the election, which, by our definition, is getting more than half of the votes (just the maximum is not enough). To make this easier, we are guaranteeing that each election has a winner, that is, there is always one candidate that wins by our definition.

Provide an algorithm to do this. **(10 points)**

21. Consider the following internetwork Π



1. Determine the number of different LANs present in Π.
2. Assign network addresses to all LANs in Π such that at least one class A, one class B, and one class C address are used.
3. Assign all hosts and router interfaces valid IP addresses.

(10 points)